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Asymptotic solutions for nonlinear magnetoconvection

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Convection in a vertical magnetic field occurs in narrow cells in the physically relevant limit where the Chandrasekhar number Q becomes large, corresponding to a strong field or small diffusion. This allows asymptotic solutions to be developed for fully nonlinear convection, requiring only the solution of a nonlinear boundary value problem. Solutions for steady and oscillatory magnetoconvection are obtained, with different scalings. In the steady case, the heat flux and the fluid velocity are found at leading order in the asymptotic expansion and the vertical velocity scales as $Q^{1/6}$. In the oscillatory case, where it is necessary to continue to second order, the vertical velocity is of order $Q^{1/3}$ and the frequency of the oscillations is always greater than that predicted by linear theory. The heat flux does not depend on either the wavenumber or the planform.

1. Introduction

Recent work by Bassom & Zhang (1994) has shown that fully nonlinear convection in a rapidly rotating layer can be described by asymptotic methods based on the narrow cell width. Here, 'fully nonlinear' means that the Rayleigh number is of the same order as the critical Rayleigh number in the asymptotic expansion, but is otherwise arbitrary. Although the fluid velocity becomes large, the flow is confined to a single horizontal wavenumber, and the only nonlinear coupling that survives is through the mean temperature. The vertical structure is determined by the solution of a single nonlinear eigenvalue problem. Some of these results were obtained by Chan (1974), but this earlier work was based on a modal truncation rather than asymptotic methods. More recently, Julien & Knobloch (1997, 1999) have extended this analysis to the cases of oscillatory and three-dimensional rotating convection, where the scalings are the same.

This paper adapts the asymptotic methods of Bassom & Zhang (1994) to the problem of convection in a vertical magnetic field. There has been much work in this area, but hitherto this has been concerned with linear and weakly nonlinear theory (Chandrasekhar 1961; Weiss 1981; Matthews & Rucklidge 1993; Clune & Knobloch 1994) or nonlinear numerical simulations (Matthews, Proctor & Weiss 1995). Magnetoconvection has been studied as an example of a dynamical system with a rich and complicated nonlinear structure, but interest in the problem originally arose from studies of sunspots. In a sunspot, convection within the Sun is inhibited by the presence of a magnetic field, causing the sunspot to appear dark (Weiss *et al.* 1996). The most important dimensionless parameters are the Chandrasekhar number Q, measuring the strength of the stabilizing magnetic field in relation to diffusion and the Rayleigh number R, measuring the ratio of the destabilizing temperature difference to diffusion. Since the diffusion terms are small on the large astrophysical scales, these parameters are both very large in a sunspot. The limit of large Q has been considered by Proctor (1986), but only in the weakly nonlinear regime.

There are two significant advantages of the present work over the the rotating case considered by Bassom & Zhang (1994). One is that rotating convection rolls are known to be unstable to the Kuppers–Lortz instability (Kuppers & Lortz 1969). In the rapidly rotating limit, Matthews & Cox (1999) have shown that the dominant instability involves rolls aligned at a small angle to the original rolls. For convection in a magnetic field there is no known analogous instability and so the solutions obtained may be stable. Secondly, the analysis here is valid for arbitrary three-dimensional flows. This is not true in the rotating case because the nonlinear terms in the momentum equation appear at leading order so the method fails. However, recent work by Julien & Knobloch (1999) shows that for certain choices of the planform function these nonlinear terms vanish and so fully nonlinear solutions can be obtained for these planforms.

Section 2 below reviews the equations, the linear stability results and the important features of the $Q \to \infty$ limit. The case of fully nonlinear steady magnetoconvection, which is a fairly straightforward adaptation of the work of Bassom & Zhang (1994), is described in §3. The oscillatory case is considered in §4; here the analysis is considerably more complicated.

2. Equations and linear theory

The dimensionless governing equations for convection in a fluid with kinematic viscosity ν in a layer of depth d with a vertical magnetic field of strength B_0 are

$$\frac{1}{\sigma}\left[\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right] = -\nabla P + R\theta\hat{\boldsymbol{z}} + \zeta Q \boldsymbol{B}\cdot\nabla\boldsymbol{B} + \nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\nabla(T+\theta) = \frac{\partial}{\partial z}\overline{w\theta} + \nabla^2\theta, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial z}\overline{w\theta} = \frac{\partial^2 T}{\partial z^2}, \tag{2.3}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \zeta\nabla^2\boldsymbol{B}. \tag{2.4}$$

The velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$ are constrained to be solenoidal. The Rayleigh number is defined by

$$R = \frac{g\alpha\Delta T d^3}{\nu\kappa}, \tag{2.5}$$

where g is the acceleration due to gravity, α is the expansion coefficient, ΔT is the temperature difference across the layer and κ is the thermal diffusivity. The field strength is measured by the Chandrasekhar number

$$Q = \frac{B_0^2 d^2}{\mu_0\rho\eta\nu}, \tag{2.6}$$

where μ_0 is the magnetic permeability, ρ is the fluid density and η is the magnetic diffusion. The remaining parameters are the Prandtl number $\sigma = \nu/\kappa$ and the magnetic Prandtl number $\zeta = \eta/\kappa$. The overbar denotes a horizontal average and the fluid temperature has been divided into a horizontally averaged part $T(z,t)$ and a fluctuating part $\theta(x,y,z,t)$.

The boundaries are stress free, fixed temperature and electrically insulating, so the components of the velocity $\boldsymbol{u} = (u,v,w)$ and magnetic field $\boldsymbol{B} = (B_x, B_y, B_z)$ obey

$$w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = B_x = B_y = \frac{\partial B_z}{\partial z} = \theta = 0 \tag{2.7}$$

at both boundaries, while $T = 0$ at the upper boundary $z = 1$ and $T = 1$ at the lower boundary $z = 0$. These boundary conditions are chosen for mathematical convenience, so that the linear eigenfunctions are trigonometric.

The stationary solution of the equations is given by $\boldsymbol{u} = \boldsymbol{0}$, $\theta = 0$, $\boldsymbol{B} = (0,0,1)$, $T = 1 - z$. The linear theory for the problem (Chandrasekhar 1961) shows that the onset of convection can be steady or oscillatory. In the steady case, the critical Rayleigh number at which the growth rate is zero is

$$R_c = \frac{(\pi^2 + k^2)^3 + Q\pi^2(\pi^2 + k^2)}{k^2}, \tag{2.8}$$

where k is the horizontal wavenumber. The wavenumber k_m that minimizes R_c obeys $2k_m^6 + 3k_m^4\pi^2 = \pi^6 + Q\pi^4$. This paper is concerned with the limit of large Q, in which case R_c also becomes large. This can be thought of as the case of a strong magnetic field, but since both R and Q are inversely proportional to the diffusion coefficients, the limit of large Q can also be thought of as the limit in which the diffusion terms become small. This is an important limit, since the diffusion terms are small in the astrophysical application. In the limit $Q \to \infty$ it follows that $k_m = O(Q^{1/6})$ and, for k of order $Q^{1/6}$,

$$\frac{R_c}{Q} = \pi^2 + O(Q^{-1/3}). \tag{2.9}$$

There are two important features of the linear theory in this limit. First, since the relevant parameter is R_c/Q, viscosity plays no role – the thermal instability is resisted by the magnetic field alone. Secondly, the marginal curve becomes flat, since at leading order R_c does not depend on k. Neither of these features hold for the case of rotating convection studied by Bassom & Zhang (1994). Because of this flattening of the marginal curve, resonant interactions between modes of different wavenumber become increasingly important at large Q, as found by Matthews *et al.* (1995).

Oscillatory convection can only occur at onset if $\zeta < 1$. For large Q the same scaling $k_m = O(Q^{1/6})$ holds and

$$R_c \sim \frac{(\sigma + \zeta)\zeta Q\pi^2}{1 + \sigma}. \tag{2.10}$$

The frequency ω of the oscillations is given by

$$\omega^2 \sim \sigma\zeta Q\pi^2(1 - \zeta)/(1 + \sigma). \tag{2.11}$$

Again the marginal curve becomes flat for large Q, but in the oscillatory case the fluid viscosity does appear, since σ occurs in (2.10) and (2.11).

3. Nonlinear steady convection

Fully nonlinear solutions to the governing equations can be obtained in the asymptotic limit $Q \to \infty$, in terms of a small parameter ϵ defined by $q = \epsilon^6 Q$, with q order one. All horizontal derivatives are of order ϵ^{-1}, while vertical derivatives remain of order one. It is also assumed that σ and ζ are order one, while $R = O(\epsilon^{-6})$. For steady convection, the appropriate scalings for fully nonlinear convection are obtained by seeking a solution in which the mean temperature T is $O(1)$. The convective heat flux $\overline{w\theta}$ must also be $O(1)$, from (2.3), and balancing the advection and diffusion terms in (2.2) gives $w = O(\epsilon^{-2}\theta)$. The scalings for velocity and temperature are therefore

$$u = O(1), \quad v = O(1), \quad w = O(\epsilon^{-1}), \quad T = O(1), \quad \theta = O(\epsilon). \tag{3.1}$$

These scalings are the same as for the case of rotating convection (Bassom & Zhang 1994). The scaling for the perturbation magnetic field, defined by $\boldsymbol{B} = (0, 0, 1) + \boldsymbol{b}$, is deduced by comparing terms in (2.4), or by balancing the buoyancy force with the Lorentz force in (2.1):

$$b_x = O(\epsilon^2), \quad b_y = O(\epsilon^2), \quad b_z = O(\epsilon). \tag{3.2}$$

This scaling is similar to that used by Proctor (1986), except that all the fluctuating quantities are larger by one power of ϵ^{-1}. A formal asymptotic expansion can be set up, but this is not necessary since all the information that is needed can be obtained at leading order. The dominant terms in the equations are then as follows. By taking the z-component of the curl of the curl of (2.1) the pressure is eliminated, giving

$$0 = -R\nabla_H^2\theta - \zeta Q \frac{\partial}{\partial z}\nabla_H^2 b_z, \tag{3.3}$$

where ∇_H^2 is the horizontal Laplacian. The leading terms in (2.2), (2.3) and the vertical component of (2.4) are

$$w\frac{\mathrm{d}T}{\mathrm{d}z} = \nabla_H^2\theta, \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\overline{w\theta} = \frac{\mathrm{d}^2T}{\mathrm{d}z^2}, \tag{3.5}$$

$$0 = \frac{\partial w}{\partial z} + \zeta\nabla_H^2 b_z. \tag{3.6}$$

It can be verified that the nonlinear terms are sufficiently small that they do not appear, except in (3.5). The equation (3.5) for the mean temperature can be integrated to give

$$\overline{w\theta} = \frac{\mathrm{d}T}{\mathrm{d}z} + N, \tag{3.7}$$

where the integration constant N is the Nusselt number, indicating the total heat flux through the layer. The equations are now separable, so that w, b_z and θ are proportional to a function $h(x, y)$ that obeys the Helmholtz equation $\nabla_H^2 h = -k^2h$. Using (3.6) and (3.4) to eliminate b_z and θ, the remaining equations (3.3) and (3.7) give

$$0 = -Rw\frac{\mathrm{d}T}{\mathrm{d}z} + Q\frac{\mathrm{d}^2w}{\mathrm{d}z^2}, \tag{3.8}$$

$$-\frac{w^2}{k^2}\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{\mathrm{d}T}{\mathrm{d}z} + N, \tag{3.9}$$

where all variables are now functions of z only and the scaling $\overline{h^2} = 1$ has been adopted. Rescaling $w = Wk$ and solving (3.9) gives

$$\frac{\mathrm{d}T}{\mathrm{d}z} = \frac{-N}{1 + W^2}, \tag{3.10}$$

so (3.8) becomes

$$\frac{d^2W}{dz^2} + \frac{RN}{Q}\frac{W}{1+W^2} = 0. \tag{3.11}$$

The value of N is determined by integrating (3.10) and applying the boundary conditions $T(0) = 1$, $T(1) = 0$:

$$N^{-1} = \int_0^1 \frac{dz}{1+W^2}. \tag{3.12}$$

The problem of nonlinear convection in a strong magnetic field has been reduced to the nonlinear eigenvalue problem given by (3.11), (3.12), with the boundary conditions $W(0) = W(1) = 0$. Note that the features of the linear problem, that there is no dependence on either the viscosity or the wavenumber, also hold in the fully nonlinear regime. Thus the magnetoconvection problem is simpler than the corresponding rotating convection problem of Bassom & Zhang (1994), where k cannot be scaled out. Another important feature that is preserved from linear theory is that the planform function $h(x, y)$ is entirely arbitrary.

It is of interest to check that linear and weakly nonlinear theory are correctly captured by (3.11), (3.12). In the linear case, $w \to 0$ so $N \to 1$ and (3.11) has the solution $W = A \sin \pi z$ with $R = Q\pi^2$. In the weakly nonlinear regime,

$$N = 1 + \int_0^1 W^2\,dz + O(A^4) \tag{3.13}$$

and so (3.11) becomes

$$\frac{Q}{R}\frac{d^2W}{dz^2} + \left(1 - W^2 + \int_0^1 W^2\,dz\right) W = 0, \tag{3.14}$$

where terms beyond cubic order in W have been dropped. Proctor & Holyer (1986) obtained (3.14) for weakly nonlinear convection in salt fingers. Writing $W = A \sin \pi z$ and then equating terms in $\sin \pi z$ in (3.14), assuming that R is near R_c, leads to the formula

$$A^2 = 4(R - R_c)/R_c \tag{3.15}$$

for the amplitude, which agrees with the weakly nonlinear result given by Weiss (1981).

Numerical solutions to (3.11), (3.12) can easily be obtained as follows. A value of RN/Q is chosen, and then (3.11) can be solved using a shooting method. The integral (3.12) is then evaluated numerically, giving the value of N and hence the corresponding value of R/Q. The results are given in figure 1 which shows the maximum vertical velocity and the Nusselt number as a function of R/Q. Profiles of vertical velocity W and mean temperature T are shown in figures 2 and 3 for three values of R/Q. In general the solution to (3.11), (3.12) is not unique, as new solutions appear at $R/Q = n^2\pi^2$ for any integer n. The solutions shown in the figures correspond to $n = 1$.

A number of analytical results can be derived from (3.11), (3.12). It is clear that $N \geqslant 1$ and that $dT/dz < 0$ throughout the layer. Integrating (3.11) once gives

$$\left(\frac{dW}{dz}\right)^2 = \frac{RN}{Q}\left(\log(1 + W_m^2) - \log(1 + W^2)\right) \tag{3.16}$$

where W_m is the maximum value of W.

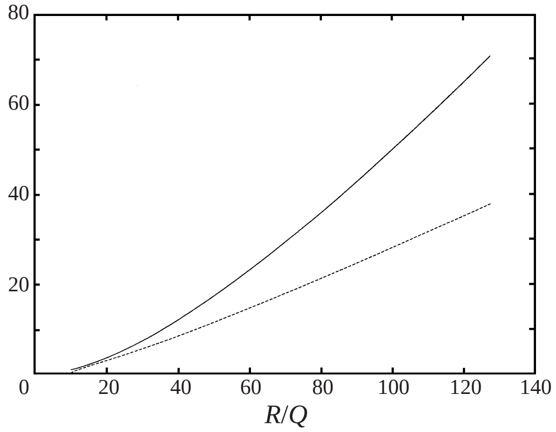

Figure 1. Nusselt number (solid line) and maximum vertical velocity (dashed line) as a function of R/Q, for steady magnetoconvection.

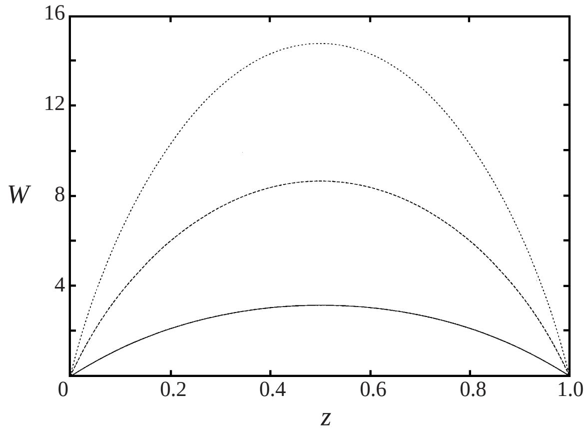

Figure 2. Profiles of vertical velocity at $R/Q = 20$ (solid line), $R/Q = 40$ (dashed line) and $R/Q = 60$ (dotted line).

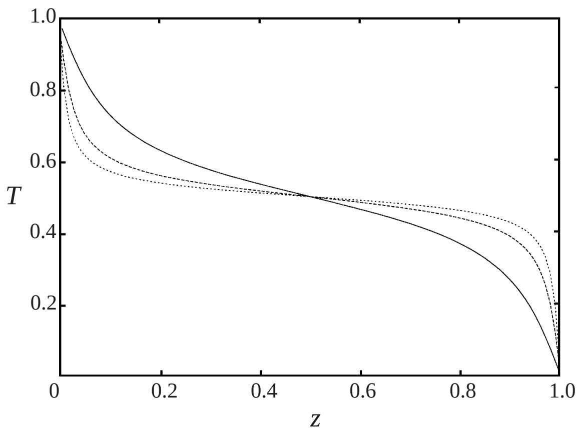

Figure 3. Temperature profiles for the same values of R/Q as in figure 2.

Further analytical progress can be made by taking the second asymptotic limit of large R/Q, when W and N also become large, as described by Bassom & Zhang (1994). The integral in (3.12) is dominated by the regions near the boundaries, and setting $W = \gamma z + \cdots$ yields $\gamma \sim N\pi$ as $\gamma \to \infty$. From (3.16) it follows that $\gamma^2 \sim 2(R/Q)N \log W_m$, so eliminating γ gives

$$N\pi^2 \sim 2(R/Q)\log W_m. \tag{3.17}$$

From (3.11) in the interior region,

$$W_m^2 = O(RN/Q), \tag{3.18}$$

and since only $\log W_m$ is needed for (3.17), an exact solution is not required. From (3.18) it follows that $2\log W_m \sim \log(R/Q) + \log N$, while (3.17) gives $\log N \sim \log(R/Q)$, so $\log W_m \sim \log(R/Q)$. Substituting for $\log W_m$ in (3.17) gives the following scaling for the Nusselt number at large R:

$$N \sim \frac{2R}{\pi^2 Q}\log(R/Q). \tag{3.19}$$

Scaling laws of this type for the analogous problem of rapidly rotating convection were derived by Chan (1974). The maximum vertical velocity W_m is of order $(R/Q)(\log(R/Q))^{1/2}$, and hence the assumption that (3.12) is dominated by the regions near the boundaries can be verified. Throughout most of the layer, $T = 0.5$, but there are boundary layers with a thickness of order $1/N$ in which $W = O(1)$ and the temperature gradient is $O(N)$.

It is important to appreciate that (3.19) cannot be valid for arbitrarily large R at fixed Q, since at sufficiently large R the magnetic field becomes dynamically unimportant and the usual scaling laws for turbulent convection, with $N = O(R^{1/3})$ or $N = O(R^{2/7})$ will apply. As R is increased, the assumption that the vertical derivatives are small compared with horizontal derivatives is no longer valid, because of the appearance of the boundary layers. As discussed by Bassom & Zhang (1994), agreement between (3.19) and the results in figure 1 is poor because the logarithmic asymptotics only becomes accurate at very large R.

4. Nonlinear oscillatory convection

In the case of oscillatory magnetoconvection a different scaling is required and the problem is more complicated as it is necessary to proceed to higher order. In the simpler case of rapidly rotating convection (Julien & Knobloch 1997), the scalings in the oscillatory case and steady case are the same and all the required information can be obtained at leading order. The difference between the two systems is that the frequency of the oscillations is $O(\epsilon^{-3})$ in magnetoconvection but $O(\epsilon^{-2})$ for rotating convection. At leading order therefore, the time-derivative terms in (2.1)–(2.4) are larger than the diffusive terms.

The appropriate asymptotic expansions are as follows:

$$w = \epsilon^{-2}w_1 + \epsilon^{-1}w_2 + \cdots, \quad T = T_1 + \epsilon T_2 + \cdots, \tag{4.1}$$

$$\theta = \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots, \quad b_z = \epsilon b_{z1} + \epsilon^2 b_{z2} + \cdots, \tag{4.2}$$

with $R = \epsilon^{-6}r$ and $Q = \epsilon^{-6}q$. The relevant fast timescale is $\tau = \epsilon^{-3}t$, suggested by the linear result (2.11); periodic solutions will be sought on this timescale. Note that the difference between the oscillatory and steady cases is that here the fluid velocity is larger by one order of ϵ^{-1}. At first sight this scaling looks incorrect because the heat flux $\overline{w\theta}$ appears to be of order ϵ^{-1}. However it turns out that this leading-order heat flux is oscillatory and hence only influences T_3.

At $O(\epsilon^{-3})$, (2.3) gives

$$\frac{\partial T_1}{\partial \tau} = 0, \tag{4.3}$$

so the mean temperature is a function of z only, $T_1 = T_1(z)$. Similarly, the $O(\epsilon^{-2})$ terms give $T_2 = T_2(z)$. At $O(\epsilon^{-1})$, (2.3) yields

$$\frac{\partial T_3}{\partial \tau} + \frac{\partial}{\partial z}\overline{w_1\theta_1} = 0. \tag{4.4}$$

Seeking a periodic solution, this equation can be averaged in time and then integrated to give

$$\langle w_1\theta_1 \rangle = 0, \tag{4.5}$$

where $\langle\ \rangle$ indicates an average over x, y and τ. At $O(1)$, (2.3) gives

$$\frac{\partial T_4}{\partial \tau} + \frac{\partial}{\partial z}\overline{w_1\theta_2 + w_2\theta_1} = \frac{d^2 T_1}{dz^2}, \tag{4.6}$$

which when averaged in time and integrated becomes

$$\langle w_1\theta_2 + w_2\theta_1 \rangle = \frac{dT_1}{dz} + N. \tag{4.7}$$

After taking the curl of the curl of (2.1), time and the horizontal dependence can be separated out, so

$$w_1 = w_1(z) \exp i\omega\tau \sum a_j \exp i\boldsymbol{k}_j \cdot \boldsymbol{x} + \text{c.c.} \tag{4.8}$$

with a wavenumber $k = |\boldsymbol{k}_j| = \epsilon^{-1}k_1$; similar expressions apply for θ_1 and b_{z1}. As in the steady case, the solution is written in terms of an arbitrary planform function. The scaling $\sum |a_j|^2 = 1$ is adopted. At next order, only those terms in w_2 and θ_2 that are proportional to $\exp i\omega\tau$ are required, since only these terms will contribute to (4.7). Thus the functions w_2, θ_2 and b_{z2} can also be written in the form (4.8). Henceforth all variables only depend on z, and (4.5) and (4.7) become

$$w_1\theta_1^* + w_1^*\theta_1 = 0, \tag{4.9}$$

$$w_1\theta_2^* + w_1^*\theta_2 + w_2\theta_1^* + w_2^*\theta_1 = \frac{dT_1}{dz} + N, \tag{4.10}$$

where * denotes the complex conjugate.

The leading terms in (2.1), (2.2), (2.4) are, at $O(\epsilon^{-5})$, $O(\epsilon^{-2})$, $O(\epsilon^{-2})$ respectively,

$$i\omega w_1/\sigma = r\theta_1 + \zeta q b'_{z1}, \tag{4.11}$$

$$i\omega\theta_1 + w_1 T'_1 = 0, \tag{4.12}$$

$$i\omega b_{z1} = w'_1, \tag{4.13}$$

where the prime denotes a z-derivative. Note from (4.12) that θ_1 and w_1 are out of phase so that (4.9) is satisfied. Eliminating θ_1 and b_{z1} leads to the following equation for w_1:

$$-\frac{\omega^2}{\sigma} w_1 = -r w_1 \frac{dT_1}{dz} + \zeta q \frac{d^2 w_1}{dz^2}. \tag{4.14}$$

The equations (2.1), (2.2), (2.4) at next order are

$$\mathrm{i}\frac{\omega}{\sigma}w_2 = r\theta_2 + \zeta q b'_{z2} - k_1^2 w_1, \tag{4.15}$$

$$\mathrm{i}\omega\theta_2 + w_2 T'_1 + w_1 T'_2 = -k_1^2\theta_1, \tag{4.16}$$

$$\mathrm{i}\omega b_{z2} = w'_2 - \zeta k_1^2 b_{z1}. \tag{4.17}$$

The nonlinear terms do not appear here, because they do not have the time and space dependence given in (4.8). Using (4.12) and (4.16) to eliminate θ_1 and θ_2, (4.10) can be evaluated. The terms in w_2 and T_2 cancel out, giving

$$T'_1 = \frac{-N}{1 + 2k_1^2|w_1|^2/\omega^2}. \tag{4.18}$$

Integrating this over the layer gives the Nusselt number:

$$N^{-1} = \int_0^1 \frac{\mathrm{d}z}{1 + 2k_1^2|w_1|^2/\omega^2}. \tag{4.19}$$

To close the problem a condition to determine ω is required. This is obtained by eliminating θ_2 and b_{z2} from (4.15)–(4.17) to give the following equation for w_2:

$$-\frac{\omega^2}{\sigma}w_2 + rw_2T'_1 - \zeta q w''_2 = \frac{k_1^2}{\mathrm{i}\omega}(rw_1T'_1 + \omega^2 w_1 - \zeta^2 q w''_1) - rw_1T'_2. \tag{4.20}$$

Since the linear operator on the left-hand side is the same as in (4.14), a solvability condition is obtained by multiplying (4.20) by w_1^* and integrating. After removing T'_1 using (4.14), the following formula for the frequency ω is obtained:

$$\omega^2(1+\sigma)\int_0^1 |w_1|^2\,\mathrm{d}z = \sigma\zeta q(1-\zeta)\int_0^1 |w'_1|^2\,\mathrm{d}z. \tag{4.21}$$

To summarize, fully nonlinear oscillatory magnetoconvection is governed by (4.19), (4.21) and (4.14) which after substituting for T'_1 using (4.18) becomes

$$\frac{\omega^2}{\sigma}w_1 + \frac{rw_1N}{1 + 2k_1^2|w_1|^2/\omega^2} + \zeta q w''_1 = 0. \tag{4.22}$$

As in the steady case, linear and weakly nonlinear theory can be checked. In the linear case, $N = 1$, (4.22) gives $w_1 = A\sin\pi z$ and $\omega^2/\sigma + r - \zeta q\pi^2 = 0$ while (4.21) yields $\omega^2(1+\sigma) = \sigma\zeta q\pi^2(1-\zeta)$. Combining these two equations, the linear results (2.10) and (2.11) are obtained. In the weakly nonlinear case, working to $O(A^2)$, (4.19) gives $N = 1 + k_1^2A^2/\omega^2$. Equating $\sin\pi z$ terms in (4.22) gives the formula for the amplitude,

$$A^2 = \frac{2\sigma(1-\zeta)(r-r_c)}{k_1^2(\sigma+\zeta)}. \tag{4.23}$$

Formulae for the amplitude of weakly nonlinear travelling waves and standing waves were given by Matthews & Rucklidge (1993); taking the limit of large Q in these formulae gives (4.23) in both cases.

Two important results follow from (4.21). The linear result that oscillatory convection can only occur for $\zeta < 1$ remains true in the nonlinear regime. Also, since

$$\int_0^1 |w'_1|^2\,\mathrm{d}z \geqslant \pi^2\int_0^1 |w_1|^2\,\mathrm{d}z \tag{4.24}$$

for any smooth w_1 obeying the boundary conditions, the nonlinear frequency of oscillation cannot be less than that given by linear theory (2.11) at the onset of convection, i.e.

$$\omega^2 \geqslant \frac{\sigma\zeta q\pi^2(1-\zeta)}{1+\sigma}. \tag{4.25}$$

This result is of interest since it is the opposite of that predicted by linear theory: computing the dependence of the frequency on R from the linearized equations, one finds that the frequency of linear oscillations decreases as R increases, and that for $R > \zeta Q\pi^2$ the growth rate is real. However, (4.25) shows that these linear results are misleading and that the fully nonlinear behaviour is quite different. Similarly, (4.25) indicates that the scenario in which nonlinear oscillations decrease in frequency until the oscillations terminate at a global bifurcation, as described by Knobloch & Weiss (1983), is not the only possibility and does not occur in this large-Q scaling.

As in the steady case, the value of the wavenumber plays no role except in the scaling of the vertical velocity. By defining $W = \sqrt{2}k_1 w_1/\omega$ the equations (4.19), (4.21) and (4.22) simplify to

$$N^{-1} = \int_0^1 \frac{\mathrm{d}z}{1+|W|^2}, \tag{4.26}$$

$$\omega^2(1+\sigma)\int_0^1 |W|^2\,\mathrm{d}z = \sigma\zeta q(1-\zeta)\int_0^1 |W'|^2\,\mathrm{d}z, \tag{4.27}$$

$$\frac{\omega^2}{\sigma}W + \frac{rWN}{1+|W|^2} + \zeta q W'' = 0, \tag{4.28}$$

so the Nusselt number N and frequency ω do not depend on the wavenumber k. Note that in the limit $\zeta \to 1$, (4.27) shows that $\omega \to 0$ and the steady problem (3.11), (3.12) is recovered.

In the limit $r \to \infty$, W also becomes large and it is of interest to determine the scaling for ω. If ω becomes large, then the leading terms in (4.28) give $\omega^2|W|^2/\sigma + rN \approx 0$, which is a contradiction since all terms are positive. Therefore the frequency ω must remain of order 1 as $r \to \infty$. The same asymptotic scaling holds as in the steady case, so that N is of order $r \log r$.

Numerical solutions to (4.26)–(4.28) can be obtained using a method similar to that used for the steady case. The only additional complication is that ω must be determined self-consistently by evaluating numerically the integrals in (4.27). The Nusselt number and the maximum vertical velocity for the case $\zeta = 0.5$, $\sigma = 1$ are shown in figure 4, and the corresponding frequency in figure 5. The frequency increases with r and approaches a constant at large r, as required by the argument above. Note however that the variation in frequency is very slight (only about 6%). This indicates that the function $W(z)$ does not deviate greatly from its initially sinusoidal form. Profiles of W and T are not plotted, as they are very similar to those for the steady case shown in figures 2 and 3.

5. Discussion

This work has generalized the asymptotic analysis for rapidly rotating convection (Bassom & Zhang 1994) to the case of convection in a strong vertical magnetic field. Prompted by linear theory, fully nonlinear solutions are obtained with a wavenumber of order $Q^{1/6}$. In the steady case, nonlinear magnetoconvection is governed by (3.11),

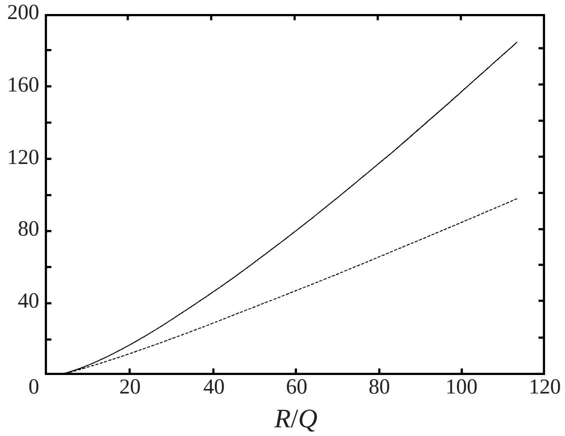

FIGURE 4. Nusselt number (solid line) and maximum vertical velocity (dashed line) as a function of R/Q, for oscillatory magnetoconvection with $\zeta = 0.5$, $\sigma = 1$.

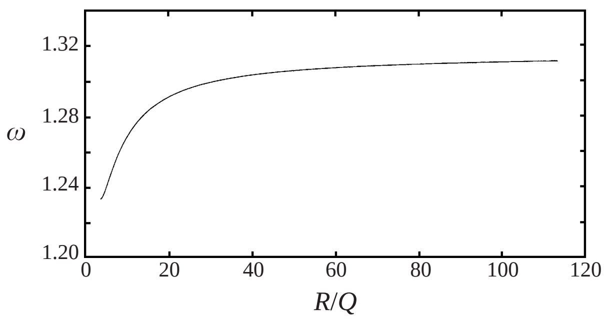

FIGURE 5. Frequency of oscillatory magnetoconvection as a function of R/Q, for $\zeta = 0.5$, $\sigma = 1$.

(3.12). It is remarkable that the heat flux N and the mean temperature profile do not depend on either of the Prandtl numbers, the planform function or even the precise value of the wavenumber k. The only remaining parameter is R/Q, so all cases are covered by figure 1. Other features of the flow are also governed by this single parameter, after scaling by an appropriate power of k. The vertical velocity is of order $Q^{1/6}$ and the horizontal velocity components are of order one. The horizontally averaged temperature is of order one, but horizontal fluctuations to the temperature and magnetic field are small.

For oscillatory magnetoconvection, the Prandtl numbers do play a role, but the heat flux and the frequency are independent of the wavenumber and the planform. In this case, the vertical and horizontal velocity components are of order $Q^{1/3}$ and $Q^{1/6}$ respectively, but despite this the flow is still dominated by a single wavenumber and the nonlinear terms are small.

Concurrent work by Julien, Knobloch & Tobias (1999) has considered the same problem but with a different scaling, where the wavenumber and the vertical velocity are of order $Q^{1/4}$. With this scaling, steady and oscillatory convection can be handled simultaneously, so that a single nonlinear eigenvalue problem is obtained covering both cases. Their results are qualitatively similar to those presented here; for example they also find that the variation of the frequency with R is small.

There are two related questions that have not been addressed by this work or by the related work on fully nonlinear rotating convection (Bassom & Zhang 1994; Julien & Knobloch 1997). The first is the question of planform selection. In principle this problem can be approached by continuing the asymptotic expansion to higher order. In fact this is a difficult task. At second order, correction terms to the flow, the temperature perturbation and the magnetic field arise, and a number of differential equations must be solved numerically to determine these. However, there is no mean temperature at this order and hence no change in the Nusselt number. To distinguish between the different possible planforms it is necessary to continue to third order and carry out further numerical computations. However, the importance of planform selection is often overestimated. Physically, the quantity of interest is the heat flux, which is determined at leading order. The second issue concerns the stability of these fully nonlinear solutions. Again, this requires higher-order effects, together with identification of the appropriate timescale. This is difficult because of the large number of different timescales in the problem, all of which must be considered if a solution is to be shown to be stable. Such complications of timescale even occur in the original linear stability analysis of the trivial state; for example when $\zeta > 1$, the growth rate is of order $Q^{1/3}$ when $Q\pi^2 < R < \zeta Q\pi^2$, but $Q^{1/2}$ when $R > \zeta Q\pi^2$.

An alternative approach to the questions of planform selection and stability is to compare the asymptotic solutions with numerical simulations of the original equations (2.1)–(2.4). This method is also not straightforward since numerical problems arise in the limit of large Q, so numerical solutions can never really reach the asymptotic limit. However, preliminary two-dimensional numerical simulations of magnetoconvection show good agreement with the asymptotic theory. Details of this work will be described in a future paper.

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